Elliptic Curves Final Examination

May 3 2023

This exam is of **50 marks** and is **3 hours long**. Please **read all the questions carefully**. Please feel free to use whatever theorems you have learned in class after stating them clearly. You may use the book '**The Arithmetic of Elliptic Curves' by Joseph Silverman**.

Please copy the following sentence on the first page of your answer sheet and write your name and signature.

I have not used any unfair or illegal means to answer any of the questions in this exam.

1. Consider the curve over $\mathbb{Q}$ given by

$$F_N : X^N + Y^N = Z^N$$

a. Show that it is a smooth curve in $\mathbb{P}^2$ and $F_1 \simeq \mathbb{P}^1$. (4)

b. Show that the map $\pi : F_N \longrightarrow F_1$ given by

$$\pi(X, Y, Z) \to [X^N, Y^N, Z^N]$$

defines a map from $\pi : F_N \to \mathbb{P}^1$ and compute its degree. (3)

c. Compute the genus of F_N. (3)

2. Consider the curve over $\mathbb{Q}$

$$F_3 : X^3 + Y^3 = Z^3$$

a. Show that together with the point $O = [1, -1, 0]$ it becomes an elliptic curve. (3)

b. Show that $P + Q + R = O \Leftrightarrow$ They are co-linear. (3)

c. What is the set of 3-torsion points defined over $\bar{\mathbb{Q}}$? (4)

3. Let E be an elliptic curve over $\mathbb{F}_q$ which is a field of characteristic p. Let $\phi : E \to E^{(q)}$ be the q^{th} power Frobenius and ϕ_ℓ is the corresponding endomorphism of the Tate module $T_\ell(E)$ where $\ell \neq p$.

a. Show that E is supersingular $\Leftrightarrow tr(\phi_\ell) = 0 \bmod p$ for any $\ell \neq p$ (5)

b. If $p \neq 2, 3$ show that E is supersingular if and only if (5)

$$|E(\mathbb{F}_p)| = p + 1$$

4. Let $\wp$ be the Weierstrass $\wp$-function and σ be the Weierstrass σ-function.

a. Show that for all $a, z \in \mathbb{C}$, (5)

$$\wp(z) - \wp(a) = -\frac{\sigma(z+a) - \sigma(z-a)}{\sigma^2(z)\sigma^2(a)}$$

b. Prove that (5)

$$\wp'(z) = -\frac{\sigma(2z)}{\sigma^4(z)}$$

5. Let E/K be an elliptic curve over a local field K with ring of integers R, maximal ideal $\mathfrak{m}$ and group of units R^*. Assume $char(K) \neq 2, 3$.

a. Let E/K be given by a Weierstrass equation

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

with $a_i \in R$. Prove that the equation is minimal if and only if either $\nu(\Delta) < 12$ or $\nu(c_4) < 4$ where ν is the valuation. (4)

b. Let E/K be given by Weierstrass equation

$$E : y^2 = X^3 + Ax + B$$

Prove that E has

1. Good reduction $\Leftrightarrow 4A^3 + 27B^2 \in R^*$ (2)
2. Multiplicative reduction $\Leftrightarrow 4A^3 + 27B^2 \in \mathfrak{m}$ and $AB \in R^*$. (2)
3. Additive reduction $\Leftrightarrow A, B \in \mathfrak{m}$ (2)